

Random collapsibility and 3-sphere recognition

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Abstract

A triangulation of a 3-manifold can be shown to be homeomorphic to the 3-sphere by describing a discrete Morse function on it with only two critical faces, that is, a sequence of elementary collapses from the triangulation with one tetrahedron removed down to a single vertex. Unfortunately, deciding whether such a sequence exist is believed to be very difficult in general.

In this article we present a method, based on uniform spanning trees, to estimate how difficult it is to collapse a given 3-sphere triangulation after removing a tetrahedron. In addition we show that out of all 3-sphere triangulations with eight vertices or less, exactly 22 admit a non-collapsing sequence onto a contractible non-collapsible 2-complex. As a side product we classify all minimal triangulations of the dunce hat, and all contractible non-collapsible 2-complexes with at most 18 triangles. This is complemented by large scale experiments on the collapsing difficulty of 9- and 10-vertex spheres.

Finally, we propose an easy-to-compute characterisation of 3-sphere triangulations which experimentally exhibit a low proportion of collapsing sequences, leading to a heuristic to produce 3-sphere triangulations with difficult combinatorial properties.

MSC 2010: 57Q15; 57N12; 57M15; 90C59. **Keywords:** Discrete Morse theory, uniform spanning trees, collapsibility, local constructibility, dunce hat, triangulated manifolds, 3-sphere, complicated triangulations

1 Introduction

Collapsibility of triangulations and closely related topics, such as local constructibility, have been studied extensively over the past decades [9, 19, 35, 38]. If a triangulation is collapsible, its underlying topological space is contractible but the converse is not true [1, 8, 10, 27, 37]. Thus, collapsibility can be seen as a measure of how complicated a triangulation of a given contractible topological space or manifold is. Understanding this complicatedness of triangulations is a central topic in the field of combinatorial topology with important consequences for theory and applications.

For instance, recognising the n -dimensional (piecewise linear standard) sphere or ball – a major challenge in the field – is still a very difficult task in dimension three and even undecidable in dimensions ≥ 5 [34]. Nonetheless, collapsing heuristics together with standard homology calculations and the Poincaré conjecture solve the n -sphere recognition problem for many complicated and large inputs in high dimensions, see for example [9, 30]. In fact, when using standard input, existing heuristics are too effective to admit proper insight into the undecidability of the underlying problem. To address this issue, Benedetti and Lutz recently proposed a “library of complicated triangulations” providing challenging input to test new methods [9].

Here we focus on the analysis of collapsibility for triangulations of the 3-ball (typically given as a 3-sphere with one tetrahedron removed). More precisely, we study this question using a

quantifying approach. Given a triangulation of the 3-ball, we analyse the question of how likely it is that a collapsing sequence of the tetrahedra, chosen uniformly at random, collapses down to a single vertex. A similar question has been studied in [9] using the framework of discrete Morse theory. While the approach in [9] is efficient, provides valuable insights, and works in much more generality (arbitrary dimension and arbitrary topology of the triangulations), the probability distributions involved in the experiments are difficult to control. As a consequence, the complicatedness of a given triangulation depends on the heuristic in use.

In this article we present an approach to quantify the “collapsing probability” of a 3-ball triangulation which can be phrased independently from the methods in use. This probability can be estimated effectively as long as there is a sufficient number of collapsing sequences. We then use this approach to give an extended study of the “collapsing probability” of small 3-sphere triangulations with one tetrahedron removed.

In addition, we decide for all 39 8-vertex 3-sphere triangulations with one tetrahedron removed whether or not they are *extendably collapsible*, that is, whether or not they have a collapsing sequence of the tetrahedra which collapses onto a contractible non-collapsible 2-dimensional complex: 17 of them are, 22 of them are not, see Theorem 4.1. As a side product of this experiment we present a classification of all minimal triangulations of the Dunce hat, cf. Theorem 4.4.

A major motivation for this study is to find new techniques to tackle the famous 3-sphere recognition problem. Recognising the 3-sphere is decidable due to Rubinstein’s algorithm [31] which has since been implemented [16, 18] and optimised by Burton [17]. However, state-of-the-art worst case running times are still exponential while the problem itself is conjectured to be polynomial time solvable [26, 32]. We believe that analysing tools – such as the ones presented in this article – and simplification procedures designed to deal with non-collapsible or nearly non-collapsible 3-sphere triangulations (i.e. input with pathological combinatorial features) together with local modifications of triangulations such as Pachner moves is one way of advancing research dealing with this important question.

Contributions

In Section 3, we describe a procedure to uniformly sample collapsing sequences in 3-ball triangulations, based on the theory of uniform spanning trees [2, 14, 25, 36].

In Section 4, we present extensive experiments on the collapsing probability of small 3-sphere triangulations with one tetrahedron removed. The experiments include a complete classification of extendably collapsible 8-vertex 3-spheres with one tetrahedron removed and a classification of 17 and 18 triangle triangulations of contractible non-collapsible 2-complexes.

In Section 5 we describe an (experimental) hint towards triangulations which are difficult to collapse. The observation translates into heuristics to generate complicated triangulations.

Software

Most of the computer experiments which have been carried out in this article can be replicated using the *GAP* package *simpcomp* [20, 21, 22, 23]. As of Version 2.1.1., *simpcomp* contains the functionality to produce *discrete Morse spectra* using the techniques developed in this article as well as the techniques from [9].

The necessary data to replicate all other experiments can be found in the appendices and/or are available from the authors upon request.

2 Preliminaries

2.1 Triangulations

Most of this work is carried out in the 3-dimensional simplicial setting. However, whenever obvious generalisations of our results and methods hold in higher dimensions, or for more general kinds of triangulations, we point this out.

By a *triangulated d -manifold* (or *triangulation of a d -manifold*) we mean a d -dimensional simplicial complex whose underlying topological space is a closed d -manifold. Note that in dimension three, the notion of a triangulated 3-manifold is equivalent to the one of a combinatorial manifold since every 3-manifold is equipped with a unique PL-structure.

A triangulation of a d -manifold M is given by a d -dimensional, pure, abstract simplicial complex C , i.e., a set of subsets $\Delta \subset \{1, \dots, v\}$ each of order $|\Delta| = d + 1$, called the *facets* of M . The i -skeleton $\text{skel}_i(M)$, that is, the set of i -dimensional faces of M can then be deduced by enumerating all subsets δ of order $|\delta| = i + 1$ which occur as a subset of some facet $\Delta \in M$. The 0-skeleton is called the *vertices* of M , denoted $V(M)$, and the 1-skeleton is referred to as the *edges* of M . The f -vector of M is defined to be $f(M) = (f_0, f_1, \dots, f_d)$ where $f_i = |\text{skel}_i(M)|$. Note that in this article we often write $f_0 = v$ and $f_d = n$, and use n as a measure of input size.

If in a triangulation M every k -tuple of vertices spans a $(k - 1)$ -face in $\text{skel}_i(M)$, i.e., if $f_{k-1} = \binom{|V(M)|}{k}$, then M is said to be *k -neighbourly*.

The *Hasse diagram* $\mathcal{H}(C)$ of a d -dimensional simplicial complex C is the directed $(d+1)$ -partite graph whose nodes are the i -faces of C , $0 \leq i \leq d$, and whose arcs point from a node representing an $(i - 1)$ -face to a node representing an i -face if and only if the $(i - 1)$ -face is contained in the i -face.

The *dual graph* or *face pairing graph* $\Gamma(M)$ of a triangulated d -manifold M is the graph whose nodes represent the facets of M , and whose arcs represent pairs of facets of M that are joined together along a common $(d - 1)$ -face. It follows that $\Gamma(M)$ is $(d + 1)$ -regular.

2.2 Uniform spanning trees and random walks

Most graphs in this article occur as the dual graph $\Gamma(M)$ of some triangulated 3-manifold M . To avoid confusion, we denote the 0- and 1-simplices of a triangulation as vertices and edges and we refer to the corresponding elements of a graph as *nodes* and *arcs*.

A *spanning tree* of a graph $G = (V, E)$ is a tree $T = (V, E')$ such that $E' \subset E$ covers all nodes in V . In other words, a spanning tree of a graph G is defined by a connected subset $E' \subset E$ of size $|E'| = |V| - 1$ such that all nodes $v \in V$ occur as an endpoint of an arc in E' . A *uniform spanning tree* $T = (V, E')$ of $G = (V, E)$ is a spanning tree chosen uniformly at random from the set of all spanning trees of G .

A *random walk of length m* in a graph $G = (V, E)$ is a sequence of random variables $(v_0, v_1, v_2, \dots, v_m)$ taking values in V , such that $v_0 \in V$ is chosen uniformly at random and for each v_i , the vertex v_{i+1} is chosen uniformly at random from all nodes adjacent to v_i in G .

2.3 Collapsibility and local constructability

Given a simplicial complex C , an i -face $\delta \in C$ is called *free* if its corresponding node in the Hasse diagram $\mathcal{H}(C)$ is of outgoing degree one. Removing a free face δ of a simplicial complex is called an *elementary collapse* of C , denoted by $C \searrow C \setminus \delta$. A simplicial complex C is called *collapsible* if there exist a sequence of elementary collapses

$$C \searrow C' \searrow C'' \searrow \dots \searrow \emptyset,$$

in this case the above sequence is referred to as a *collapsing sequence* of C (sometimes we omit the last elementary collapse from a single vertex to the empty set and still refer to the sequence as a collapsing sequence).

If, for a simplicial complex C , *every* sequence of removing free faces leads to a collapsing sequence, C is called *extendably collapsible*. If, on the other hand, no collapsing sequence exist, C is said to be *non-collapsible*.

Given a d -dimensional simplicial complex C , we say that C is *locally constructible* or that C *admits a local construction*, if there is a sequence of pure simplicial complexes $T_1, \dots, T_n, \dots, T_N$ such that (i) T_1 is a d -simplex, (ii) T_{i+1} , $i+1 \leq n$, is constructed from T_i by gluing a new tetrahedron to T_i along one of its $(d-1)$ -dimensional boundary faces, (iii) T_{i+1} , $i+1 > n$, is constructed from T_i by identifying a pair of $(d-1)$ faces of T_i whose intersection contains a common $(d-2)$ -dimensional face, and (iv) $T_N = C$.

For $d = 3$, locally constructible spheres were introduced by Durhuus and Jonsson in [19]. Locally constructible triangulations of 3-spheres are precisely the ones which are collapsible after removing a facet due to a result by Benedetti and Ziegler [10].

For the remainder of this article we sometimes call a triangulated 3-sphere S *collapsible* if it is locally constructible, i.e., if there exist a facet $\Delta \in S$ such that $S \setminus \Delta$ is collapsible. This notion is independent of the choice of Δ (cf. [10, Corollary 2.11]). The idea behind this abuse of the notion of collapsibility is to refer to those 3-sphere triangulations as collapsible which have a chance of being recognised by a collapsing heuristic.

3 Collapsibility of 2-complexes and uniform spanning trees

In this section, we want to propose a method to quantify collapsibility of 3-sphere triangulations (with one tetrahedron removed). Deciding collapsibility is hard in general but easy in most cases which occur in practice and thus methods to measure the degree to which a triangulation is collapsible are of great help in the search for pathological, i.e., non-collapsible 3-ball triangulations. The idea is closely related to the concept of the discrete Morse spectrum as presented in [9], the main difference being that our method is *independent* of the collapsing heuristic in use. This, however, comes at the cost of only focusing on triangulations of the 3-sphere and possibly slight generalisations thereof.

Our method uses the facts that (i) collapsibility of arbitrary 2-complexes is easy to decide by a linear time greedy type algorithm [33, Proposition 5], (ii) spanning trees of a graph can efficiently be sampled uniformly at random (see below for more details), and (iii) the process of collapsing the 3-cells of a 3-manifold triangulation M along a spanning tree of the dual graph T is well defined. That is, we can collapse all 3-cells of M along T by first removing the 3-cell $\Delta \in M$ corresponding to the root node of T and then successively collapse all other 3-cells through the 2-cells of M corresponding to the arcs of T , and this procedure does not depend on the choice of Δ , see [10, Corollary 2.11].

More precisely, for a 3-sphere triangulation S , we can efficiently sample a spanning tree in the dual graph $T \subset \Gamma(S)$, collapse all 3-cells of S along T , and then decide collapsibility of the remaining 2-complex in linear time in the number of facets of S . Our method leads to the following notion.

Definition 3.1 (Collapsing probability). Let S be a 3-sphere triangulation and let $p \in [0, 1]$ be a (rational) number between zero and one. We say that S has *collapsing probability* p if the number of spanning trees leading to a collapsing sequence of S divided by the total number of spanning trees of S equals p .

In particular, collapsing probability 0 is equivalent to non-collapsibility and collapsing probability 1 is equivalent to extendable collapsibility.

The above definition corresponds to a shortened version of what is called the *discrete Morse spectrum* in [9].

Given the notion of collapsing probability, the potential difficulty of deciding collapsibility for a 3-sphere triangulation S (with one tetrahedron removed) must be entirely encapsulated within its extremely large number of possible spanning trees: if this number were small, we could simply try all spanning trees of S until we either find a collapsing sequence or conclude that S (with one tetrahedron removed) is non-collapsible. But how many spanning trees of $\Gamma(S)$ exist?

The dual graph of any 3-manifold triangulation is 4-regular. Hence, following [29] the number of spanning trees of a 3-sphere triangulation with n tetrahedra is bounded above by

$$\# \text{ spanning trees} < C \cdot \left(\frac{27}{8}\right)^n \frac{\log n}{n} < \frac{9}{2} \left(\frac{27}{8}\right)^n.$$

Thus, enumerating spanning trees to decide collapsibility does not seem like a viable option.

However, the related task of sampling a spanning tree uniformly at random is efficiently solvable. The first polynomial time algorithm to uniformly sample spanning trees in an arbitrary graph was presented by Guénoche in 1983 [25]. It has a running time of $O(n^3 m)$ where n is the number of nodes and m is the number of arcs of the graph. Hence, in the case of the 4-valent dual graph of a triangulated 3-manifold, the running time of Guénoche's algorithm is $O(n^4)$. Since then many more deterministic algorithms were constructed with considerably faster running times. Here, we want to consider a randomised approach. Randomised sampling algorithms for spanning trees were first presented by Broder [14] and Aldous [2]. Their approach is based on a simple idea using random walks. Given a graph G , follow a random walk in G until all nodes have been visited discarding all arcs on the way which close a cycle. The result can be shown to be a spanning tree chosen uniformly at random amongst *all* spanning trees of the graph. The expected running time equals what is called the *cover time* of G , i.e., the expected time it takes a random walk to visit all nodes in G , with a worst case expected running time of $O(n^3)$. For many graphs, however, the expected running time is as low as $O(n \log n)$. The algorithm we want to use for our purposes is an improvement of the random walk construction due to Wilson [36] which always beats the cover time. More precisely, the expected running time of Wilson's algorithm is $O(\tau)$ where τ denotes the expected number of steps of a random walk until it hits an already determined subtree $T' \subset G$ starting from a node which is not yet covered by T' .

Observation 3.2. Let S be a 3-sphere triangulation with n tetrahedra and collapsing probability $p \in [0, 1]$. Sampling a uniform spanning tree in the dual graph of S and testing collapsibility of the remaining 2-complex is a Bernoulli trial

$$X = \begin{cases} 1 & \text{with probability } p; \\ 0 & \text{else.} \end{cases}$$

with polynomial running time. Sampling N times yields N such independent Bernoulli distributed random variables X_i , $1 \leq i \leq N$, and the maximum likelihood estimator

$$\hat{p} = \frac{1}{N} \sum_{i=1}^N X_i$$

follows a normalised Binomial distribution with $\mathbb{E}\hat{p} = p$ and $\text{Var } \hat{p} = p(1-p)/N$. By Chebyshev's inequality this translates to

$$\mathbb{P}(|\hat{p} - p| \leq \epsilon) < \frac{p(1-p)}{N\epsilon^2} \leq \frac{1}{4N\epsilon^2}.$$

Since we want to decide collapsibility of S (with one tetrahedron removed), we want to distinguish p from 0. Thus, setting $\epsilon = p/2$ we get

$$\mathbb{P}(|\hat{p} - p| \leq p/2) < \frac{4(1-p)}{Np}$$

and for $p \rightarrow 0$ the error can be controlled by N being (super-)linear in p^{-1} .

Altogether, we note that collapsibility of S (with one tetrahedron removed) can be rejected with a high level of confidence by a polynomial procedure as long as p^{-1} is polynomial in the size of the input n .

Note that, using Wilson's algorithm, every sampling procedure only depends on the size of the triangulation by a factor which, in average, has running time less than $O(n \log n)$. This makes the approach well-suited for computer experiments on larger triangulations where a higher proportion of triangulations with a low collapsing rate is conjectured.

4 Estimated collapsing probabilities for small 3-sphere triangulations

A small triangulation of a 3-sphere admits very few or no spanning trees in the dual graph, which lead to non-collapsing sequences. As the size of triangulations increase, we expect that the number of non-collapsible sequences increases as well. See [30] for experiments supporting this claim in higher dimensions. However, given a sequence of somewhat "averagely complicated" 3-sphere triangulations in increasing size, the question of how exactly the *proportion* of collapsing sequences to non-collapsing sequences changes, that is, how quickly (if at all) the collapsing probability p decreases, is an interesting question with deep implications for important problems in the field of computational 3-manifold topology. One major difficulty in this context comes from the fact that it is not at all clear what is meant by an "averagely complicated" 3-sphere triangulation.

In this section, we use the method of uniformly sampling spanning trees described above, together with the classification of 3-sphere triangulations up to ten vertices to get a more thorough overview in the case of known small triangulations.

3-spheres with up to 8-vertices

It is well-known that all 3-balls with seven or less vertices are extendably collapsible [5]. Hence, all v -vertex 3-sphere triangulations with $v \leq 7$ must have collapsing probability 1.

There are 39 distinct 8-vertex triangulations of the 3-sphere, first listed by Grünbaum and Sreedharan in [24]. We use their notation for the remainder of this section. Two of them are non-polytopal, one of them known as *Barnette's sphere* [6], the other one known as *Brückner's sphere* (see [24] where the sphere, first described in [15], is first shown to be non-polytopal).

The collapsibility of these 3-sphere triangulations (with one tetrahedron removed) was studied in [7], where the authors showed that three of the 39 spheres contain a collapsing sequence onto a dunce hat and hence have collapsing probability < 1 .

Here, we refine this study by a complete description of 8-vertex 3-spheres with collapsing probability 1 and a list of non-perfect Morse functions for the remaining cases. Namely, we present a computer assisted proof of the following statement.

Theorem 4.1. *There are 17 8-vertex triangulations of the 3-sphere which, after removing any tetrahedron, are extendably collapsible. In the Grünbaum, Sreedharan notation these are*

$$P_1, P_2, P_3, P_4, P_5, P_6, P_7, P_9, P_{10}, P_{13}, P_{14}, P_{15}, P_{16}, P_{17}, P_{18}, P_{21}, P_{34}.$$

The remaining 22 8-vertex triangulations of the 3-sphere triangulations admit a collapsing sequence onto a contractible non-collapsible 2-complex.

In order to construct a computer assisted proof for Theorem 4.1, we first need to make some theoretical observations.

Lemma 4.2. *Let C be a contractible non-collapsible simplicial 2-complex. Then C must have at least 8 vertices and 17 triangles.*

Proof. Let C be a contractible, non-collapsible 2-complex of minimal size. Since C is contractible, we must have vanishing 2-homology. In particular $H_2(C, \mathbb{F}_2) = 0$, where \mathbb{F}_2 is the field with two elements. Look at the formal sum σ of all triangles of C . σ is a 2-chain and its boundary $\partial\sigma$ contains all edges of C of odd degree.

If C has no edges of odd degree then σ is a non-vanishing element of $H_2(C, \mathbb{F}_2)$, contradiction. Hence, C must have edges of odd degree. In addition, since the edges of odd degree are a boundary, they must form a closed cycle and since C is simplicial, this cycle must be of length at least three.

Since C is minimal non-collapsible, no edge can be of degree one, and altogether all edges must be of degree at least two and at least three edges must be of degree at least three.

Let $f(C) = (n, f_1, f_2)$ be the f -vector of C . Since C is contractible, C has Euler characteristic one. Hence

$$\begin{aligned} 1 &= n - f_1 + f_2 \\ f_2 &= \frac{1}{3} \sum_{e \text{ edge of } C} \deg(e), \end{aligned}$$

and f_2 is minimal if and only if three edges are of degree three and all other edges are of degree two. Inserting these degrees into the second equation yields

$$f_2 \geq \frac{2}{3}f_1 + 1$$

and using the first equation we get

$$f_2 \geq 2n + 1.$$

Following the results in [4] we have $n \geq 8$ and thus C must have at least 17 triangles. \square

Corollary 4.3. *Let S be a 3-sphere triangulation with f -vector $f(S) = (v, v + n, 2n, n)$, and let B be a 3-ball obtained from S by removing a tetrahedron. If $v < 8$ or $n < 16$ then B is extendably collapsible.*

Proof. A 3-ball B is extendably collapsible if, after removing its 3-cells along a spanning tree of the dual graph, the remaining contractible 2-complex must be collapsible.

After collapsing the tetrahedra of S along a spanning tree the remaining 2-complex has $(n + 1)$ triangles and v vertices. The result now follows from Lemma 4.2. \square

The observation made in Corollary 4.3 can be extended to 3-sphere triangulations S with a larger number $n \geq 16$ of tetrahedra. If we can show that the 2-skeleton of S does not contain a contractible non-collapsible 2-complex with at most $n + 1$ triangles, then S (with one tetrahedron removed) must be extendably collapsible. In order for this approach to work, we need to know all such 2-complexes up to $n + 1$ triangles. Like any other attempt to exhaustively classify small triangulations, this rapidly becomes infeasible with n growing larger. However, in the border case of $16 \leq n \leq 17$ this task turns out to be well in reach.

Theorem 4.4. *The only non-collapsible contractible simplicial complexes with 8 vertices and 17 triangles are the seven minimal triangulations of the dunce hat shown in Figure 4.1. Furthermore, the only non-collapsible contractible simplicial complexes with 8 vertices and 18 triangles are the 19 minimal saw-blade complexes with four, three, and two blades shown in Figures 4.2, 4.3, and 4.4 respectively, and the 61 triangulations of the Dunce hat listed in Appendix B.*

Proof. Since $n \leq 18$, no edges of degree other than two and three can exist, and the edges of degree three must form a simple cycle. By ungluing the edges of degree three, an 8-vertex, 17-triangle non-collapsible contractible 2-complex can thus be represented as a 14-vertex triangulation of the disk with a nine-gon as boundary plus some boundary identifications. Analogously, an 8-vertex, 18-triangle non-collapsible contractible 2-complex can be represented as a 16-vertex triangulation of the disk with a 12-gon as boundary and some boundary identifications.

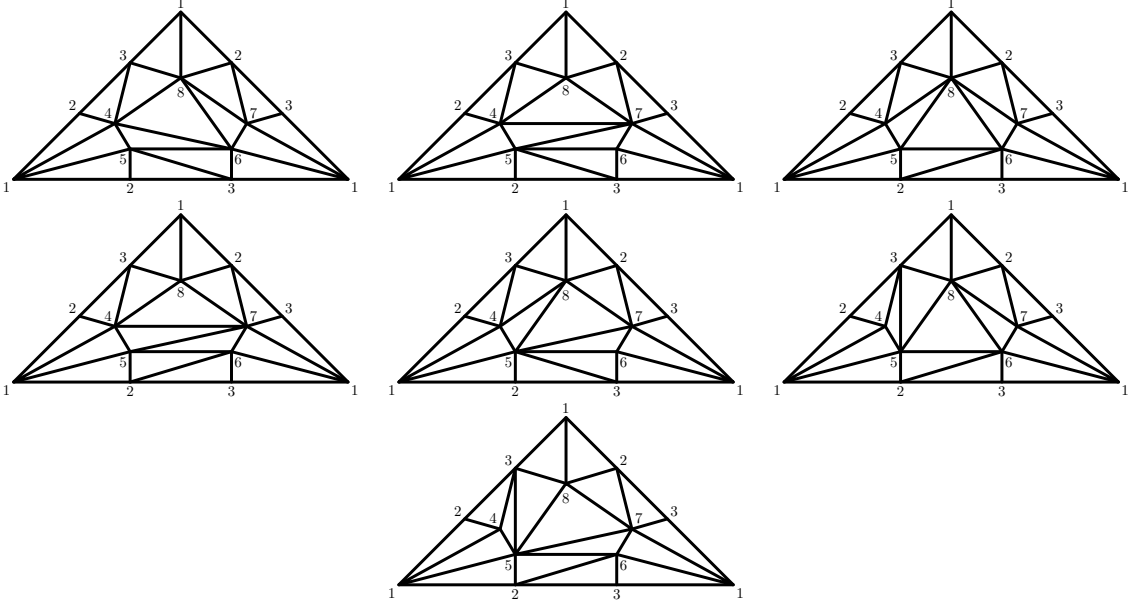


Figure 4.1: A classification of minimal triangulations of the Dunce hat.

Using the software for planar graph enumeration by Brinkmann and McKay [12, 13] together with its enormously useful plug-in framework, we classify all such disks, only keeping those which can be folded up to result in an 8-vertex 17 (18) triangle contractible non-collapsible simplicial complex with 21 degree two and 3 (4) degree three edges. Sorting out isomorphic copies yields seven minimal triangulations of the dunce hat (drawn in Figure 4.1), 61 dunce hats with 18 triangles, and 19 examples of three distinct types of so-called *saw-blade complexes*, cf. [30] (see Figures 4.2, 4.3, and 4.4 for the 19 saw blade complexes and Appendix B for a list of all 18-triangle complexes). \square

With these results in place we can now describe a computer assisted proof of Theorem 4.1.

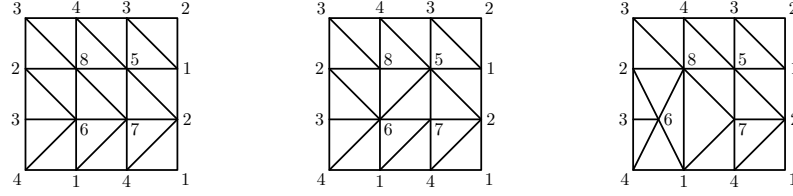


Figure 4.2: A classification of minimal saw-blade complexes with four blades.

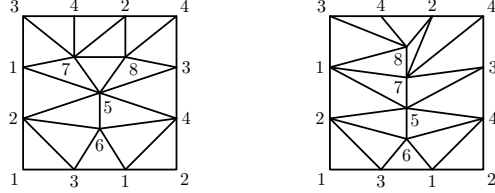


Figure 4.3: A classification of minimal saw-blade complexes with three blades.

Proof of 4.1. Eight of the 39 8-vertex 3-spheres have 15 or less tetrahedra and thus, by Corollary 4.3, must have collapsing probability 1 (i.e., they are extendably collapsible after removing a tetrahedron). In Grünbaum and Sreedharan's labelling these are the triangulations P_1 to P_7 , and P_{13} .

Furthermore, using our uniform sampling technique of spanning trees we are able to collapse 22 of the remaining 31 triangulations to a contractible non-collapsible 2-complex and thus show that these have collapsing probability < 1 , i.e., that none of them, after removing a tetrahedron, is extendably collapsible. A certificate for the non-extendable collapsibility for each of the 22 cases, in form of a non-perfect Morse function, can be found in Appendix A.

The remaining nine cases, triangulations $P_9, P_{10}, P_{14}, \dots, P_{18}, P_{21}, P_{34}$ in [24], have between 16 and 17 tetrahedra. Following the proof of Corollary 4.3, after collapsing the tetrahedra of an n -tetrahedra 8-vertex 3-sphere S along a spanning tree the remaining 2-complex C , which by construction must be contractible, has $(n + 1)$ triangles and at most 8 vertices. Combining Lemma 4.2 and Theorem 4.4 this means that C either collapses onto a point or it is isomorphic to one of the seven minimal triangulations of the dunce hat, for $16 \leq n \leq 17$, or it is isomorphic to one of the 80 contractible non-collapsible 2-complexes with 18 triangles, in the case of $n = 17$ only.

An exhaustive search for all labellings of all 87 complexes in the 2-skeleta of all nine 3-spheres showed that none of the nine remaining 8-vertex 3-spheres contain such a contractible non-collapsing 2-complex. Thus all of them are, after removing a tetrahedron, extendably collapsible. \square

In order to complement the qualitative study on collapsibility given by Theorem 4.1, we applied our uniform spanning tree heuristic to the 19 and 20 tetrahedra 8-vertex 3-spheres using 10^7 samples for each complex. This gives a more detailed estimate for the collapsing probability in the 8-vertex case.

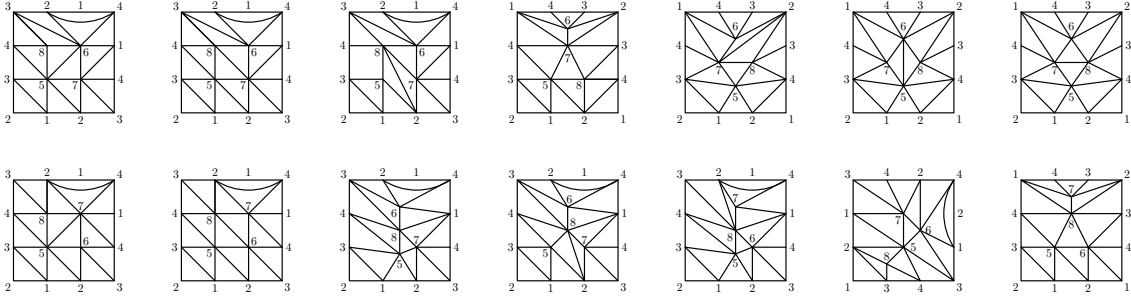


Figure 4.4: A classification of minimal saw-blade complexes with two blades.

The results are listed in Table 1 together with their edge variance, which will be discussed as indicator for the collapsing probability in Section 5.

Table 1: Estimated collapsing probabilities and edge variance of 8-vertex 3-sphere triangulations with 19 and 20 tetrahedra, sample size $N = 10^7$.

name in [24]	f -vector / isomorphism signature*	coll. prob.	edge var.
	(8, 27, 38, 19)		
B	deeffag.hbg.hag.hbg.hchdgbh.hehgh**	99.99902%	0.83951
P_{32}	deeffaf.gbg.gbgbh.gbh.hch.hahghcg	99.99922%	0.83951
P_{31}	deeffaf.gbg.gbgbh.hbg.hch.hghbhjh	99.99984%	0.98765
P_{30}	deeffaf.gbg.gbhbh.hbh.gfhahchehgg	99.99990%	0.98765
P_{33}	deeffaf.gbh.hhbhg.gbg.gbhcgegag.g	99.99988%	1.13580
	(8, 28, 40, 20)		
M	deeffaf.gbh.gbgbh.gbh.hch.hbg.gehcg***	99.99657%	0.91837
P_{37}	deeffaf.gbg.gbgbh.hbh.hch.hahehah.h	99.99942%	1.20408
P_{36}	deeffaf.gbg.gbhbh.hbh.hchbahcheh.h	99.99968%	1.20408
P_{35}	deeffaf.gbg.gbhbh.hbh.gdhahahchfgfg	99.99997%	1.34694

* The isomorphism signature of a combinatorial manifold uniquely determines its isomorphism type, i.e., two combinatorial manifolds have equal isomorphism signature if and only if they are isomorphic. The isomorphism signature given in this table coincides with the one used by simpcomp [20, 21, 22]. Use the function `SCFromIsoSig(...)` to generate the complexes. See the manual for details.

** This is Barnette's sphere.

*** This is Brückner's sphere.

9-vertex spheres

There are 1296 triangulated 9-vertex 3-spheres first described in [3]. We sampled $5 \cdot 10^4$ spanning trees for each of them. The results are summarised below where all 3-sphere triangulations with equal number of tetrahedra n are grouped together. Similar to the 8-vertex case, a higher number of tetrahedra correlates with a lower collapsing probability. The rightmost column shows the complex with the smallest collapsing probability for each class of triangulations. Note that some of the empirical collapsing probabilities below are too close to 1 in order to give robust estimators.

Table 2: Estimated collapsing probabilities of 9-vertex 3-sphere triangulations grouped by f -vector, and minimal estimated(!) collapsing probability per group, sample size $N = 5 \cdot 10^4$ per complex.

n	# complexes	avg. coll. prob.	min. coll. prob.
17	7	100.00000%	100.000%
18	23	100.00000%	100.000%
19	45	100.00000%	100.000%
20	84	99.99993%	99.998%
21	128	99.99983%	99.998%
22	175	99.99952%	99.996%
23	223	99.99898%	99.994%
24	231	99.99753%	99.980%
25	209	99.99443%	99.962%
26	121	99.99051%	99.952%
27	50	99.98024%	99.920%

In Section 5 below we discuss how the square of the average difference between an edge degree and the average edge degree of a triangulation, the *edge variance* (cf. Definition 5.2), influences the collapsing probability of a triangulation. Essentially, the findings of Section 5 suggest that a smaller edge variance correlates with a lower collapsing probability. The following table lists empirical collapsing probabilities for the triangulation with minimum edge variance for each class of triangulations with fixed f -vector with a much higher number of 10^6 samples per complex. Compare the estimated collapsing probabilities with the values from the table above and with the values given for the 8- and 10-vertex case.

Table 3: Estimated collapsing probabilities of 9-vertex, n -tetrahedron 3-sphere triangulations with minimum edge variance, $23 \leq n \leq 27$, sample size $N = 10^6$ per complex.

n	isomorphism signature*	edge degrees	coll. prob.
23	deeffag.hbg.iag.ibh.ichbidh.ibhbi.hfipijh	$3^5 4^{14} 5^{11} 6^2$	99.9981%
24	deeffaf.gbh.gbgbi.gbh.hch.ibg.geicg.hgigiji	$3^6 4^{13} 5^{10} 6^4$	99.9912%
25	deeffaf.gbh.gbgbi.gbh.ici.ibi.gbibhchcikhdg	$3^6 4^{12} 5^{12} 6^4$	99.9766%
26	deeffaf.gbh.gbgbh.gbh.ici.hbi.ibibhaiag.ifihhjg	$3^6 4^{13} 5^{11} 6^4 7^1$	99.9485%
27	deeffaf.gbh.gbgbi.gbh.ici.hbi.ibibhaiahahahcihhjg	$3^6 4^{12} 5^{15} 7^3$	99.9007%

The full list of complexes and their empirical collapsing probabilities for $5 \cdot 10^4$ samples is available from the authors upon request.

10-vertex spheres

There are 247882 triangulated 9-vertex 3-spheres first described in [28]. We sampled $5 \cdot 10^3$ spanning trees for each of them. The results grouped by number of tetrahedra are summarised in the table below.

Table 4: Estimated collapsing probabilities of 10-vertex 3-sphere triangulations grouped by f -vector, and minimal estimated(!) collapsing probability per group, sample size $N = 5 \cdot 10^3$ per complex.

n	# complexes	avg. coll. prob.	min. coll. prob.
20	30	100.00000%	100.00%
21	124	100.00000%	100.00%
22	385	99.99990%	99.98%
23	952	99.99989%	99.98%
24	2142	99.99966%	99.96%
25	4340	99.99936%	99.96%
26	8106	99.99860%	99.94%
27	13853	99.99750%	99.92%
28	21702	99.99521%	99.90%
29	30526	99.99144%	99.86%
30	38553	99.98578%	99.80%
31	42498	99.97656%	99.72%
32	39299	99.96899%	99.52%
33	28087	99.94089%	99.40%
34	13745	99.91159%	99.16%
35	3540	99.87571%	99.20%

Again, for each number of tetrahedra, we ran 10^6 samples on the triangulation with minimum edge variance.

Table 5: Estimated collapsing probability of 10-vertex, n -tetrahedron 3-sphere triangulation with minimum edge variance, $27 \leq n \leq 35$, sample size $N = 10^6$ per complex.

n	isomorphism signature*	edge degrees	coll. prob.
27	deeffaf.gbhbibj.ibh.jcj.jbg.iciahcigjghcidiijjki	$3^4 4^{18} 5^{12} 6^3$	99.9974%
28	deeffaf.gbhbibj.ibh.hch.jbg.iciajci.hgjgjbidiibjwi	$3^4 4^{19} 5^{10} 6^5$	99.9917%
29	deefgaf.hbg.hbi.iai.j.iaj.hdiaj.j.ibj.jajchckjhkgjdirh	$3^5 4^{16} 5^{13} 6^5$	99.9739%
30	deeffag.hbi.jag.ibh.jbi.ibj.ichchbj.hbj.jbgcgdjchcgjDh	$3^{10} 5^{30}$	99.9612%
31	deeffaf.gbhbibj.ibh.hbi.i.hbi.jeg.ibgcgdgdfjcjcbgcitj	$3^7 4^{10} 5^{19} 6^5$	99.8560%
32	deefgaf.hbg.ibj.jah.i.jag.jbi.jci.jeigf.jbfgi.hbicj.fjhdhah	$3^5 4^{14} 5^{18} 6^4 7^1$	99.7397%
33	deefgaf.hbg.hbi.iah.j.iag.icicj.ibj.jajgf.ibfgj.hbjcjhghfdh.h	$3^5 4^{15} 5^{16} 6^6 7^1$	99.5714%
34	deefgaf.hbi.gbhb.iahaiag.jbj.jbg.g.hafcg.ibjcf.jbjgcdigjfhjbccc	$3^6 4^{12} 5^{19} 6^6 7^1$	99.2457%
35	deefgaf.hbi.gbhb.hajajai.hbgaibj.hbjbeiafaiciaj.gbjcj.ijghg.ggi.i	$3^5 4^{17} 5^{16} 6^2 7^5$	99.1755%

The full list of complexes and their empirical collapsing probabilities for $5 \cdot 10^3$ samples is available from the authors upon request.

Altogether, if we restrict ourselves to v -vertex 2-neighborly 3-sphere triangulations, $v \leq 10$, we have for the average empirical collapsing probability

v	# complexes	1- avg. coll. prob.
≤ 7	3	0.0000000
8	4	0.0000109
9	50	0.0002064
10	3540	0.0012429

This very limited amount of data already exhibits a rapid increase in the proportion of non-collapsing sequences for triangulations of increasing size. This supports speculations that pathologically complicated combinatorial objects become rapidly more common as the size of a triangulation increases. However, whether or not these numbers actually suggests that the average collapsing probability approaches 0 as $v \rightarrow \infty$ remains unclear.

5 Producing nearly non-collapsible 3-sphere triangulations

In this section, we propose a heuristic to pre-evaluate if a 3-sphere triangulation has complicated combinatorial properties (such as very few or no collapsing sequences) or not, based on the simple combinatorial property of the edge degrees of the triangulation. Of course, there must be theoretical limits to how effective this pre-evaluation can be due to the potential hardness of the underlying problem, but a deeper understanding of the impact of simple, local combinatorial structures on (non-)collapsing sequences gives valuable insights into triangulations with pathological combinatorial characteristics.

Let S be a triangulation of the 3-sphere with f -vector $f(S) = (v, v+n, 2n, n)$ and for any edge $e \in \text{skel}_1(S)$ let $\deg_S(e)$ be the edge degree of e in S (i.e., the number of tetrahedra of S containing e). Furthermore, let $\Gamma(S)$ be the face pairing graph of S and let $T \subset \Gamma(S)$ be a spanning tree. The 2-dimensional simplicial complex obtained by collapsing S along T is denoted by S_T , and e is called *free* in S_T if e has degree one in S_T (i.e., it is only contained in one triangle of S_T).

For any given spanning tree $T \subset \Gamma(S)$, we expect the number of free edges in S_T (more precisely, the number of triangles with free edges) to strongly correlate with whether or not T leads to a collapsing sequence: triangles of S_T can be removed as long as there are free edges left and the removal of any triangle has a clear tendency to produce new free edges. The more free edges there are to begin with, the higher we expect the chances to be that in this process all triangles can be removed.

In more concrete terms, let $\text{skel}_1(S) = \{e_1, \dots, e_{n+v}\}$ be the set of edges of S and let

$$p_i = |\{e_i \text{ free in } S_T \mid T \text{ spanning tree of } S\}| / |\{\text{spanning trees of } S\}|$$

be the probability that edge e_i is free in S_T for T sampled uniformly at random. Then, the vector (p_1, \dots, p_{n+v}) contains information about the collapsing probability of S .

Obviously, the exact value of the p_i depends on the structure of S and getting a precise estimate for the p_i involves sampling spanning trees (which, at the same time, gives us an estimator for the collapsing probability – the quantity we want to pre-evaluate). Instead we argue that the degree of the edge e_i , a quantity which can be very easily extracted from S , influences the value of p_i by virtue of the following observation.

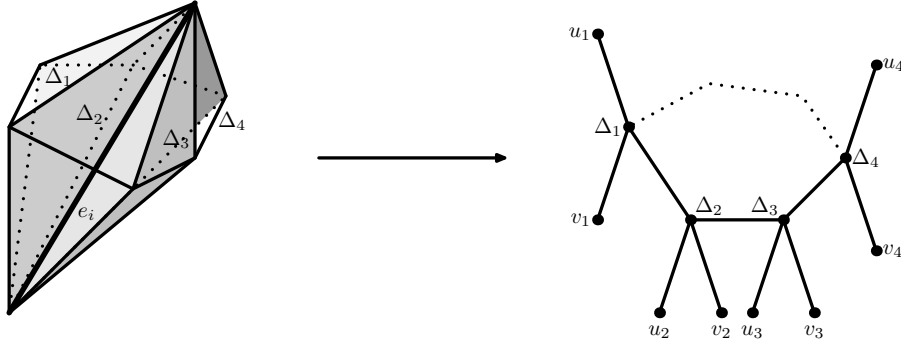


Figure 5.1: Left: star of an edge in a simplicial 3-dimensional triangulation. Right: the corresponding feature in the dual graph.

Theorem 5.1. *Let S be a triangulation of the 3-sphere and let $e_i \in \text{skel}_1(S)$ be an edge of S of degree $\deg_S(e_i) = k$. Furthermore, let p_i be the proportion of spanning trees of $\Gamma(S)$ for which e_i is free in S_T . Then*

$$p_i \geq \frac{4}{7} \cdot \left(\frac{4}{13} \right)^{k-2}.$$

Proof. Since S is a simplicial complex, the star of e_i is represented in $\Gamma(S)$ as the boundary of a k -gon $C = \langle \Delta_1, \Delta_2, \dots, \Delta_k \rangle$ with no chords. To see this note that a chord in the k -gon represents an identification of triangular boundary faces of the star of e_i in S which contradicts the simplicial complex property (cf. Figure 5.1 on the left hand side). Moreover, for any spanning tree $T \subset \Gamma(S)$, e_i is free in S_T if and only if T intersects C in a (connected) path of length $k-1$.

A spanning tree $T \subset \Gamma(S)$ can be found by following a random walk in $\Gamma(S)$ discarding all arcs on the way which close a cycle. Here, we only concentrate on the probability of one particular class of random walks which always result in a 2-complex S_T in which e_i is free.

W.l.o.g., let Δ_1 be the first node in $\Gamma(S)$ which is visited by the random walk in step m . One way for $T \cap C$ to be a path of length $k-1$ is if the random walk travels through all arcs (Δ_j, Δ_{j+1}) , $1 \leq j \leq k-1$, travelling back and forth between nodes Δ_j and u_j, v_j or Δ_{j-1} on the way (cf. Figure 5.1 on the right hand side). In step $m+1$ the walk is at one of Δ_2 or Δ_k with probability $\frac{1}{2}$. If the random walk does not choose one of these two options, there is an overall $\frac{1}{8}$ chance that it revisits Δ_1 in step $m+2$ without visiting any of the Δ_ℓ first (remember: there are no chords in the k -cycle). Moreover, there is an at least $\frac{1}{64}$ chance that it revisits Δ_1 in step $m+4$, etc. Altogether, there is a

$$\sum_{i \geq 0} 8^{-i} = \frac{4}{7}$$

chance that the random walk hits Δ_2 or Δ_k such that it is still in the class of random walks we are considering (and, in particular, can still produce a spanning tree leading to a free edge e_i).

W.l.o.g., let the random walk be now at Δ_2 (if it is at Δ_k we relabel). There is a $\frac{1}{4}$ chance that the random walk is at Δ_3 in the next step and a $\frac{3}{16}$ chance that the random walk revisits Δ_2 after two steps (without hitting any node other than u_2, v_2 or Δ_1). Hence, there is an overall chance of

$$\frac{1}{4} \cdot \sum_{i \geq 0} \left(\frac{3}{16} \right)^i = \frac{4}{13}$$

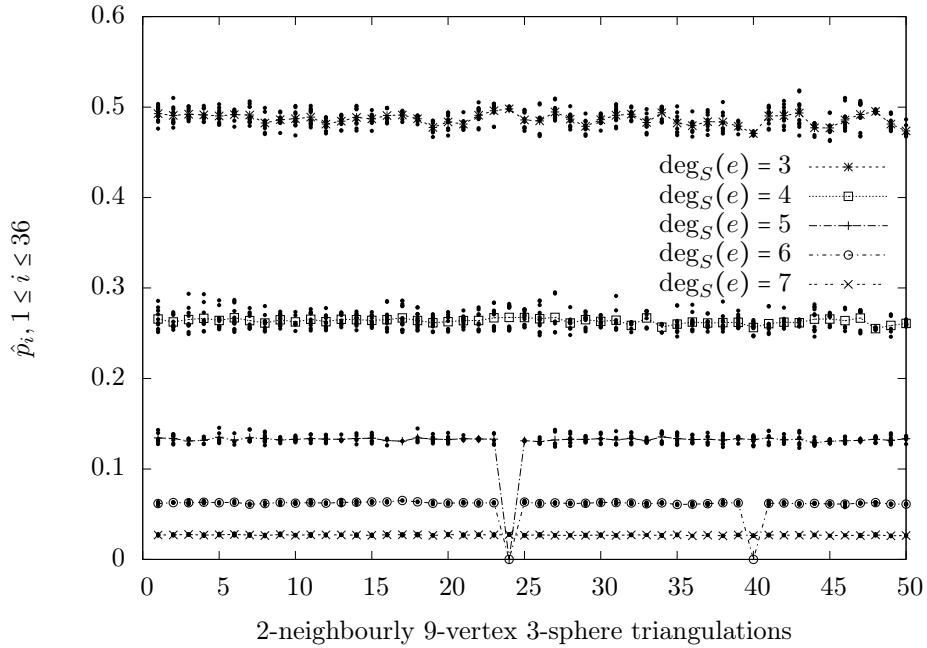


Figure 5.2: Every dot represents the estimated probability \hat{p}_i of an edge e_i , $1 \leq i \leq 36$, in one of the 2-neighbourly 9-vertex 3-sphere triangulations being free after collapsing the 3-cells along a uniformly sampled spanning tree. The lines indicate the average probability of an edge being free grouped by degree. Sample size 10^5 .

that the random walk reaches Δ_3 and still is in the class of random walks we are considering (and, in particular, still has the chance to produce a spanning tree leading to a free edge e_i). Iterating the argument proves the theorem. \square

The above lower bounds are not expected to be sharp, especially not for higher degrees. However, finding effective upper bounds for the p_i 's is difficult due to the unknown global structure of S . Nonetheless, Theorem 5.1 supports the intuitive assumption that edges of small degrees have a much higher chance of becoming free than higher degree edges.

To see how the lower bounds given in Theorem 5.1 compare to the actual values of the p_i we take a closer look at the set of 2-neighbourly 9-vertex 3-sphere triangulations. Namely, for every 9-vertex 2-neighbourly 3-sphere S and for every edge $e_i \in S$, $1 \leq i \leq n + v$, we uniformly sample 10^5 spanning trees T and record how often e_i is a free edge in S_T – the 2-complex obtained by collapsing all tetrahedra of S along T . We then compare this number to the degree $\deg_S(e_i)$ of e_i .

The results of this experiment are shown in Figure 5.2. On the horizontal axis all 50 2-neighbourly 9-vertex triangulations of the 3-sphere are listed. The vertical axis lists the estimators \hat{p}_i , $1 \leq i \leq n + v$, of the probability of the edge e_i to be free in S_T with spanning tree T chosen uniformly at random. Note how the estimators \hat{p}_i display an exponential decay in the degree of the edges, as suggested by Theorem 5.1. Moreover, for most edges e_i of degree $\deg_S(e_i)$, we have for the estimator $\hat{p}_i \sim 2^{2-\deg_S(e_i)}$.

The edge variance of a triangulation

Assuming the above observations about the quantities p_i hold in reasonable generality this motivates the following strategy to produce complicated triangulations of the 3-sphere (i.e., triangulations of the 3-sphere with collapsing probability near zero).

Let S be an n -tetrahedra v -vertex triangulation of the 3-sphere. Every tetrahedron has six edges and the total number of edges in S is given by $n + v$. Hence, the average edge degree of an edge in S is given by

$$\overline{\deg}_S = \frac{6n}{n + v}.$$

In addition, since S is simplicial, every edge degree must be at least three.

We have seen in the above section that, given a spanning tree $T \subset \Gamma(S)$ chosen uniformly at random, the value p_i for edge e_i being of degree k can be assumed to decay exponentially in k . At the same time, it is reasonable to assume that, at least on average, collapsing sequences.

Combining these two statements, we can expect that 3-sphere triangulations with few edges of degree three and four should be, on average, more difficult to collapse than triangulations with many low degree edges. To quantify this property we define the following combinatorial invariant.

Definition 5.2. Let S be an n -tetrahedron, v -vertex simplicial triangulation of the 3-sphere, and let $\overline{\deg}_S = \frac{6n}{n+v}$ be its average edge degree. The quantity

$$\text{var}(S) = \frac{1}{n + v} \sum_{i=1}^{n+v} (\overline{\deg}_S - \deg_S(e_i))^2$$

is referred to as the *edge variance* of S .

Given a 3-sphere triangulation S , computing $\text{var}(S)$ is a very simple procedure but might give away valuable hints towards the collapsing probability of S .

Note that this measure must fail in general since non-collapsibility (i.e., collapsing probability zero) can be a local feature of a triangulation and thus cannot always be picked up by the edge variance (which has a global characteristic). Nonetheless, first experiments with heuristics based on the edge variance seem promising (see below).

Remark 5.3. For experimental evidence that the edge variance is indeed a valuable measure of complicatedness, compare the average collapsing probability of 8-, 9- and 10-vertex 3-sphere in Tables 1, 3 and 5, and the collapsing probabilities for spheres with minimal edge variance in Tables 1, 2, and 4 respectively. Keep in mind that some of the complexes admit very few non-collapsing sequences and thus only the estimators of near-neighbourly triangulations can be assumed to be reasonably robust.

A heuristic to produce complicated 3-sphere triangulations

The heuristic to produce complicated 3-sphere triangulations is straightforward: Given a 3-sphere triangulation S , we perform bistellar one- and two-moves in order to reduce the edge variance. The heuristic follows a simulated annealing approach where phases of reducing the edge variance are followed by phases where the edge variance is deliberately increased. For a much more detailed discussion of simulated annealing type simplification heuristics based on bistellar moves see [11]. The complex with currently the smallest edge variance is stored and returned after a maximum number of moves is performed.

As a proof of concept, we are able to produce a 15-vertex 3-sphere S_{15} triangulation with only $2.5903 \pm 0.0618\%$ collapsing sequences, error probability 0.01%. Its facet list is given in Appendix C. See the table below for a comparison of this number with known small and complicated 3-sphere triangulations from [9].

triangulation	f -vector	exp. coll. prob. ($N = 10^6$)
S_{15}	(15, 105, 180, 90)	0.025903 ± 0.000618
trefoil	(13, 69, 112, 56)	0.839725 ± 0.001427
double_trefoil	(16, 108, 184, 92)	0.193914 ± 0.001538
triple_trefoil	(18, 143, 250, 125)	0.000000 ± 0.000000

Extending this approach to a more sophisticated heuristic with optimised parameters is work in progress.

The potential of this approach lies in applying this framework to the inverse problem of producing a collapsible triangulation. Given a 3-manifold triangulation M , one of the most basic approaches to prove that M is a 3-sphere is to describe a collapsing sequence of M . This approach, however, cannot work for non-collapsible 3-sphere triangulations. Using the idea of the edge variance, and given a complicated triangulation of a 3-manifold, we first try to increase its edge variance and then try to collapse it.

Implementing such a heuristic is simple. However, testing its effectiveness is not: as of today there are simply not enough small but complicated 3-spheres known to test this approach against existing heuristics.

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A Non-collapsing sequences of 22 of the 39 3-sphere triangulations with 8-vertices

Table 6: Non-collapsing sequences (non-perfect discrete Morse functions) of 22 of the 39 8-vertex 3-sphere triangulations.

name	facets	collapsing of 3-skeleton	contr. non-coll. 2-compl.
P_{25}	$\{1, 2, 3, 4\}, \{1, 3, 6, 8\}, \{1, 2, 3, 6\},$ $\{2, 3, 5, 6\}, \{3, 5, 6, 8\}, \{4, 5, 6, 8\},$ $\{3, 5, 7, 8\}, \{4, 5, 7, 8\}, \{4, 6, 7, 8\},$ $\{1, 4, 5, 7\}, \{1, 4, 6, 7\}, \{1, 2, 4, 6\},$ $\{1, 6, 7, 8\}, \{1, 3, 7, 8\}, \{1, 3, 5, 7\},$ $\{1, 3, 4, 5\}, \{2, 3, 4, 5\}, \{2, 4, 5, 6\}.$	$\{3, 4, 5\} \rightarrow \{3, 4, 5, 4\}, \{2, 4, 5, 6\}, \{2, 4, 5\} \rightarrow \{2, 4, 5, 4\},$ $\{1, 3, 7\} \rightarrow \{1, 3, 7, 7\}, \{1, 3, 5, 7\} \rightarrow \{1, 3, 5, 5\},$ $\{2, 4, 6\} \rightarrow \{2, 4, 6, 4\}, \{1, 4, 6\} \rightarrow \{1, 4, 6, 6\},$ $\{1, 4, 7\} \rightarrow \{1, 4, 7, 5\}, \{6, 7, 8\} \rightarrow \{6, 7, 8, 7\},$ $\{4, 5, 7\} \rightarrow \{4, 5, 7, 7\}, \{5, 7, 8\} \rightarrow \{5, 7, 8, 7\},$ $\{4, 5, 8\} \rightarrow \{4, 5, 8, 6\}, \{5, 6, 8\} \rightarrow \{5, 6, 8, 6\},$ $\{3, 5, 6\} \rightarrow \{3, 5, 6, 5\}, \{2, 3, 6\} \rightarrow \{2, 3, 6, 3\},$ $\{1, 3, 8\} \rightarrow \{1, 3, 8, 6\}, \{1, 2, 3\} \rightarrow \{1, 2, 3, 3\}.$	$\{4, 6, 7\}, \{4, 7, 8\}, \{4, 6, 8\},$ $\{1, 3, 6\}, \{1, 3, 4\}, \{3, 6, 8\},$ $\{1, 6, 7\}, \{3, 7, 8\}, \{2, 3, 4\},$ $\{1, 2, 4\}, \{3, 5, 7\}, \{2, 3, 5\},$ $\{1, 2, 6\}, \{2, 5, 6\}, \{4, 5, 6\},$ $\{1, 4, 5\}, \{1, 5, 7\}.$
P_{32}	$\{1, 4, 6, 7\}, \{1, 2, 3, 4\}, \{1, 3, 4, 5\},$ $\{2, 5, 7, 8\}, \{1, 2, 4, 6\}, \{4, 5, 6, 8\},$ $\{2, 4, 5, 6\}, \{2, 5, 6, 8\}, \{2, 3, 6, 8\},$ $\{1, 2, 3, 6\}, \{1, 3, 6, 7\}, \{1, 3, 5, 7\},$ $\{1, 4, 5, 7\}, \{4, 5, 7, 8\}, \{2, 3, 4, 5\},$ $\{4, 6, 7, 8\}, \{3, 6, 7, 8\}, \{2, 3, 7, 8\},$ $\{2, 3, 5, 7\}.$	$\{3, 7, 8\} \rightarrow \{3, 7, 8, 7\}, \{2, 3, 5, 7\}, \{2, 3, 7\} \rightarrow \{2, 3, 7, 7\},$ $\{2, 3, 5\} \rightarrow \{2, 3, 5, 4\}, \{4, 7, 8\} \rightarrow \{4, 7, 8, 7\},$ $\{4, 5, 7\} \rightarrow \{4, 5, 7, 5\}, \{1, 5, 7\} \rightarrow \{1, 5, 7, 5\},$ $\{1, 3, 7\} \rightarrow \{1, 3, 7, 6\}, \{1, 3, 6\} \rightarrow \{1, 3, 6, 3\},$ $\{2, 3, 6\} \rightarrow \{2, 3, 6, 6\}, \{2, 6, 8\} \rightarrow \{2, 6, 8, 6\},$ $\{2, 4, 5\} \rightarrow \{2, 4, 5, 5\}, \{4, 5, 6\} \rightarrow \{4, 5, 6, 6\},$ $\{2, 4, 6\} \rightarrow \{2, 4, 6, 4\}, \{2, 5, 8\} \rightarrow \{2, 5, 8, 7\},$ $\{3, 4, 5\} \rightarrow \{3, 4, 5, 4\}, \{1, 3, 4\} \rightarrow \{1, 3, 4, 3\},$ $\{1, 4, 6\} \rightarrow \{1, 4, 6, 6\}.$	$\{4, 6, 8\}, \{4, 6, 7\}, \{1, 4, 7\},$ $\{1, 6, 7\}, \{4, 5, 8\}, \{1, 2, 6\},$ $\{3, 6, 8\}, \{1, 4, 5\}, \{1, 3, 5\},$ $\{3, 6, 7\}, \{3, 5, 7\}, \{5, 6, 8\},$ $\{2, 5, 6\}, \{5, 7, 8\}, \{1, 2, 3\},$ $\{2, 3, 8\}, \{2, 5, 7\}, \{2, 7, 8\}.$
P_{31}	$\{2, 5, 7, 8\}, \{5, 6, 7, 8\}, \{4, 5, 6, 7\},$ $\{1, 2, 4, 6\}, \{1, 4, 6, 7\}, \{1, 6, 7, 8\},$ $\{1, 3, 6, 8\}, \{1, 4, 5, 7\}, \{1, 3, 4, 5\},$ $\{1, 2, 3, 4\}, \{1, 2, 3, 6\}, \{2, 3, 6, 8\},$ $\{2, 5, 6, 8\}, \{2, 4, 5, 6\}, \{2, 3, 4, 5\},$ $\{2, 3, 5, 7\}, \{2, 3, 7, 8\}, \{1, 3, 7, 8\},$ $\{1, 3, 5, 7\}.$	$\{3, 7, 8\} \rightarrow \{3, 7, 8, 7\}, \{1, 3, 5, 7\}, \{1, 3, 7\} \rightarrow \{1, 3, 7, 7\},$ $\{2, 3, 5\} \rightarrow \{2, 3, 5, 4\}, \{2, 4, 5\} \rightarrow \{2, 4, 5, 5\},$ $\{2, 5, 6\} \rightarrow \{2, 5, 6, 6\}, \{2, 6, 8\} \rightarrow \{2, 6, 8, 6\},$ $\{2, 3, 6\} \rightarrow \{2, 3, 6, 3\}, \{1, 2, 3\} \rightarrow \{1, 2, 3, 3\},$ $\{1, 3, 4\} \rightarrow \{1, 3, 4, 4\}, \{1, 4, 5\} \rightarrow \{1, 4, 5, 5\},$ $\{1, 3, 6\} \rightarrow \{1, 3, 6, 6\}, \{1, 7, 8\} \rightarrow \{1, 7, 8, 7\},$ $\{1, 6, 7\} \rightarrow \{1, 6, 7, 6\}, \{1, 4, 6\} \rightarrow \{1, 4, 6, 4\},$ $\{4, 5, 7\} \rightarrow \{4, 5, 7, 6\}, \{5, 6, 7\} \rightarrow \{5, 6, 7, 7\},$ $\{5, 7, 8\} \rightarrow \{5, 7, 8, 7\}.$	$\{2, 3, 8\}, \{1, 3, 8\}, \{1, 6, 8\},$ $\{1, 3, 5\}, \{1, 2, 6\}, \{3, 4, 5\},$ $\{2, 3, 4\}, \{2, 4, 6\}, \{1, 2, 4\},$ $\{4, 6, 7\}, \{6, 7, 8\}, \{1, 4, 7\},$ $\{4, 5, 6\}, \{1, 5, 7\}, \{2, 7, 8\},$ $\{5, 6, 8\}, \{2, 5, 7\}, \{2, 5, 8\}.$
P_{37}	$\{2, 4, 5, 6\}, \{2, 3, 5, 7\}, \{1, 3, 6, 8\},$ $\{1, 6, 7, 8\}, \{1, 2, 4, 6\}, \{1, 4, 6, 7\},$ $\{1, 3, 5, 7\}, \{1, 3, 7, 8\}, \{2, 3, 7, 8\},$ $\{2, 5, 7, 8\}, \{4, 5, 7, 8\}, \{1, 4, 5, 7\},$ $\{1, 3, 4, 5\}, \{2, 3, 4, 5\}, \{1, 2, 3, 4\},$ $\{1, 2, 3, 6\}, \{2, 3, 6, 8\}, \{2, 5, 6, 8\},$ $\{4, 5, 6, 8\}, \{4, 6, 7, 8\}.$	$\{5, 6, 8\} \rightarrow \{5, 6, 8, 6\}, \{4, 6, 8\} \rightarrow \{4, 6, 8, 6\},$ $\{2, 3, 6\} \rightarrow \{2, 3, 6, 3\}, \{1, 2, 3\} \rightarrow \{1, 2, 3, 3\},$ $\{2, 3, 4\} \rightarrow \{2, 3, 4, 4\}, \{1, 3, 4\} \rightarrow \{1, 3, 4, 4\},$ $\{1, 4, 5\} \rightarrow \{1, 4, 5, 5\}, \{4, 5, 7\} \rightarrow \{4, 5, 7, 7\},$ $\{5, 7, 8\} \rightarrow \{5, 7, 8, 7\}, \{2, 7, 8\} \rightarrow \{2, 7, 8, 7\},$ $\{3, 7, 8\} \rightarrow \{3, 7, 8, 7\}, \{1, 3, 7\} \rightarrow \{1, 3, 7, 5\},$ $\{4, 6, 7\} \rightarrow \{4, 6, 7, 6\}, \{1, 4, 6\} \rightarrow \{1, 4, 6, 4\},$ $\{1, 7, 8\} \rightarrow \{1, 7, 8, 7\}, \{1, 6, 8\} \rightarrow \{1, 6, 8, 6\},$ $\{2, 3, 5\} \rightarrow \{2, 3, 5, 5\}, \{2, 4, 5\} \rightarrow \{2, 4, 5, 5\}.$	$\{2, 4, 6\}, \{2, 5, 6\}, \{1, 2, 4\},$ $\{1, 4, 7\}, \{4, 7, 8\}, \{4, 5, 6\},$ $\{2, 5, 7\}, \{4, 5, 8\}, \{1, 5, 7\},$ $\{1, 2, 6\}, \{2, 3, 7\}, \{2, 5, 8\},$ $\{3, 5, 7\}, \{2, 3, 8\}, \{1, 6, 7\},$ $\{6, 7, 8\}, \{3, 6, 8\}, \{1, 3, 5\},$ $\{1, 3, 6\}.$
P_{30}	$\{1, 3, 7, 8\}, \{1, 2, 4, 6\}, \{2, 3, 4, 5\},$ $\{1, 3, 6, 8\}, \{1, 4, 6, 8\}, \{1, 4, 7, 8\},$ $\{4, 5, 7, 8\}, \{2, 5, 6, 7\}, \{2, 3, 5, 7\},$ $\{1, 3, 5, 7\}, \{2, 4, 5, 6\}, \{1, 4, 5, 7\},$ $\{1, 3, 4, 5\}, \{1, 2, 3, 4\}, \{1, 2, 3, 6\},$ $\{2, 3, 6, 7\}, \{3, 6, 7, 8\}, \{5, 6, 7, 8\},$ $\{4, 5, 6, 8\}.$	$\{6, 7, 8\} \rightarrow \{6, 7, 8, 7\}, \{4, 5, 6, 8\}, \{5, 6, 8\} \rightarrow \{5, 6, 8, 7\},$ $\{2, 3, 6\} \rightarrow \{2, 3, 6, 3\}, \{3, 6, 7\} \rightarrow \{3, 6, 7, 6\},$ $\{1, 3, 4\} \rightarrow \{1, 3, 4, 4\}, \{1, 2, 3\} \rightarrow \{1, 2, 3, 3\},$ $\{4, 5, 6\} \rightarrow \{4, 5, 6, 5\}, \{1, 4, 5\} \rightarrow \{1, 4, 5, 5\},$ $\{3, 5, 7\} \rightarrow \{3, 5, 7, 5\}, \{1, 5, 7\} \rightarrow \{1, 5, 7, 5\},$ $\{4, 5, 7\} \rightarrow \{4, 5, 7, 7\}, \{2, 5, 7\} \rightarrow \{2, 5, 7, 6\},$ $\{1, 4, 8\} \rightarrow \{1, 4, 8, 6\}, \{4, 7, 8\} \rightarrow \{4, 7, 8, 7\},$ $\{2, 4, 5\} \rightarrow \{2, 4, 5, 4\}, \{1, 6, 8\} \rightarrow \{1, 6, 8, 6\},$ $\{3, 7, 8\} \rightarrow \{3, 7, 8, 7\}, \{2, 4, 6\} \rightarrow \{2, 4, 6, 4\}.$	$\{1, 2, 6\}, \{2, 3, 4\}, \{2, 5, 6\},$ $\{1, 3, 6\}, \{2, 6, 7\}, \{5, 6, 7\},$ $\{1, 2, 4\}, \{1, 3, 8\}, \{2, 3, 7\},$ $\{1, 3, 7\}, \{1, 7, 8\}, \{3, 6, 8\},$ $\{1, 4, 6\}, \{2, 3, 5\}, \{5, 7, 8\},$ $\{4, 6, 8\}, \{3, 4, 5\}, \{4, 5, 8\}.$
P_{36}	$\{3, 6, 7, 8\}, \{2, 5, 7, 8\}, \{1, 4, 5, 7\},$ $\{1, 4, 6, 8\}, \{1, 2, 4, 6\}, \{1, 2, 3, 4\},$ $\{2, 6, 7, 8\}, \{2, 5, 6, 8\}, \{2, 3, 4, 5\},$ $\{2, 4, 5, 6\}, \{4, 5, 6, 8\}, \{4, 5, 7, 8\},$ $\{1, 4, 7, 8\}, \{1, 3, 7, 8\}, \{1, 3, 6, 8\},$ $\{1, 2, 3, 6\}, \{2, 3, 6, 7\}, \{2, 3, 5, 7\},$ $\{1, 3, 5, 7\}, \{1, 3, 4, 5\}.$	$\{3, 5, 7\} \rightarrow \{1, 3, 4, 5\}, \{1, 3, 5\} \rightarrow \{1, 3, 5, 5\},$ $\{2, 3, 6\} \rightarrow \{3, 5, 7, 5\}, \{2, 3, 7\} \rightarrow \{2, 3, 7, 6\},$ $\{1, 3, 8\} \rightarrow \{2, 3, 6, 3\}, \{1, 3, 6\} \rightarrow \{1, 3, 6, 6\},$ $\{4, 7, 8\} \rightarrow \{1, 3, 8, 7\}, \{1, 7, 8\} \rightarrow \{1, 7, 8, 7\},$ $\{4, 5, 6\} \rightarrow \{4, 7, 8, 7\}, \{4, 5, 8\} \rightarrow \{4, 5, 8, 6\},$ $\{2, 5, 6\} \rightarrow \{4, 5, 6, 5\}, \{2, 4, 5\} \rightarrow \{2, 4, 5, 4\},$ $\{2, 5, 6\} \rightarrow \{2, 5, 6, 6\}, \{2, 6, 8\} \rightarrow \{2, 6, 8, 7\},$ $\{1, 3, 4\} \rightarrow \{1, 3, 4, 3\}, \{1, 2, 4\} \rightarrow \{1, 2, 4, 4\},$ $\{1, 4, 6\} \rightarrow \{1, 4, 6, 6\}, \{1, 5, 7\} \rightarrow \{1, 5, 7, 5\},$ $\{2, 5, 7\} \rightarrow \{2, 5, 7, 7\}, \{6, 7, 8\} \rightarrow \{6, 7, 8, 7\}.$	$\{3, 6, 7\}, \{3, 7, 8\}, \{3, 6, 8\},$ $\{1, 6, 8\}, \{2, 7, 8\}, \{2, 3, 5\},$ $\{2, 5, 8\}, \{2, 6, 7\}, \{1, 2, 6\},$ $\{1, 2, 3\}, \{1, 3, 7\}, \{1, 4, 8\},$ $\{1, 4, 7\}, \{4, 6, 8\}, \{5, 7, 8\},$ $\{4, 5, 7\}, \{2, 4, 6\}, \{3, 4, 5\},$ $\{2, 3, 4\}.$
M	$\{1, 3, 4, 5\}, \{1, 4, 5, 8\}, \{1, 2, 3, 4\},$ $\{3, 6, 7, 8\}, \{4, 6, 7, 8\}, \{4, 5, 6, 8\},$ $\{2, 4, 5, 6\}, \{2, 3, 4, 5\}, \{2, 3, 7, 8\},$ $\{1, 4, 6, 7\}, \{1, 2, 4, 6\}, \{2, 3, 5, 7\},$ $\{1, 3, 5, 7\}, \{1, 3, 6, 7\}, \{1, 2, 3, 6\},$ $\{2, 3, 6, 8\}, \{2, 5, 6, 8\}, \{2, 5, 7, 8\},$ $\{1, 5, 7, 8\}, \{1, 4, 7, 8\}.$	$\{5, 7, 8\} \rightarrow \{1, 4, 7, 8\}, \{1, 7, 8\} \rightarrow \{1, 7, 8, 7\},$ $\{2, 6, 8\} \rightarrow \{5, 7, 8, 7\}, \{2, 5, 8\} \rightarrow \{2, 5, 8, 6\},$ $\{1, 3, 6\} \rightarrow \{2, 6, 8, 6\}, \{2, 3, 6\} \rightarrow \{2, 3, 6, 3\},$ $\{3, 5, 7\} \rightarrow \{1, 3, 6, 6\}, \{1, 3, 7\} \rightarrow \{1, 3, 7, 5\},$ $\{1, 4, 6\} \rightarrow \{3, 5, 7, 5\}, \{1, 2, 6\} \rightarrow \{1, 2, 6, 4\},$ $\{2, 3, 5\} \rightarrow \{1, 4, 6, 6\}, \{2, 3, 7\} \rightarrow \{2, 3, 7, 7\},$ $\{2, 3, 5\} \rightarrow \{2, 3, 5, 4\}, \{2, 4, 5\} \rightarrow \{2, 4, 5, 5\},$ $\{4, 5, 6\} \rightarrow \{4, 5, 6, 6\}, \{4, 6, 8\} \rightarrow \{4, 6, 8, 7\},$ $\{6, 7, 8\} \rightarrow \{6, 7, 8, 7\}, \{2, 3, 4\} \rightarrow \{2, 3, 4, 3\},$ $\{1, 4, 8\} \rightarrow \{1, 4, 8, 5\}, \{1, 4, 5\} \rightarrow \{1, 4, 5, 4\}.$	$\{3, 7, 8\}, \{2, 7, 8\}, \{2, 5, 7\},$ $\{3, 6, 7\}, \{3, 6, 8\}, \{1, 5, 7\},$ $\{1, 4, 7\}, \{2, 3, 8\}, \{4, 6, 7\},$ $\{1, 2, 3\}, \{1, 2, 4\}, \{2, 4, 6\},$ $\{1, 3, 5\}, \{2, 5, 6\}, \{1, 3, 4\},$ $\{3, 4, 5\}, \{5, 6, 8\}, \{4, 5, 8\},$ $\{4, 7, 8\}.$

continued on next page –

C A complicated 15-vertex triangulation of the 3-sphere

f -vector: (15, 105, 180, 90)

Collapsing probability: $2.5903 \pm 0.0618\%$ (for error probability 0.01%)

Isomorphism signature:

```
deefgaf.hbi.gbj.kalamai.nbo.n.lbl.oaj.hbg.k.oaoamaj.ibi.hbl.hbm.obg
.l.kam.jcicn.mak.nbkakcmak.lahanbm.nbg.mclcmbncn.jbnchambkbccc.jbfa
kaf.fcifkbiekgmbchikm.nbnfboddf.iecsjan0d
```

$\langle 1234 \rangle,$	$\langle 1237 \rangle,$	$\langle 1246 \rangle,$	$\langle 12611 \rangle,$	$\langle 12713 \rangle,$
$\langle 121113 \rangle,$	$\langle 1345 \rangle,$	$\langle 1357 \rangle,$	$\langle 1459 \rangle,$	$\langle 14610 \rangle,$
$\langle 14915 \rangle,$	$\langle 141015 \rangle,$	$\langle 1578 \rangle,$	$\langle 15813 \rangle,$	$\langle 15912 \rangle,$
$\langle 151213 \rangle,$	$\langle 161011 \rangle,$	$\langle 17813 \rangle,$	$\langle 191214 \rangle,$	$\langle 191415 \rangle,$
$\langle 1101114 \rangle,$	$\langle 1101415 \rangle,$	$\langle 1111213 \rangle,$	$\langle 1111214 \rangle,$	$\langle 2345 \rangle,$
$\langle 2358 \rangle,$	$\langle 23712 \rangle,$	$\langle 23812 \rangle,$	$\langle 2456 \rangle,$	$\langle 25614 \rangle,$
$\langle 25814 \rangle,$	$\langle 261115 \rangle,$	$\langle 261415 \rangle,$	$\langle 27912 \rangle,$	$\langle 27913 \rangle,$
$\langle 28910 \rangle,$	$\langle 28912 \rangle,$	$\langle 281014 \rangle,$	$\langle 291013 \rangle,$	$\langle 2101315 \rangle,$
$\langle 2101415 \rangle,$	$\langle 2111315 \rangle,$	$\langle 35710 \rangle,$	$\langle 35815 \rangle,$	$\langle 351011 \rangle,$
$\langle 351115 \rangle,$	$\langle 361213 \rangle,$	$\langle 361215 \rangle,$	$\langle 361314 \rangle,$	$\langle 361415 \rangle,$
$\langle 371012 \rangle,$	$\langle 381215 \rangle,$	$\langle 391011 \rangle,$	$\langle 391013 \rangle,$	$\langle 391115 \rangle,$
$\langle 391314 \rangle,$	$\langle 391415 \rangle,$	$\langle 3101213 \rangle,$	$\langle 4569 \rangle,$	$\langle 4678 \rangle,$
$\langle 46710 \rangle,$	$\langle 4689 \rangle,$	$\langle 47814 \rangle,$	$\langle 471012 \rangle,$	$\langle 471214 \rangle,$
$\langle 48911 \rangle,$	$\langle 481114 \rangle,$	$\langle 491115 \rangle,$	$\langle 4101213 \rangle,$	$\langle 4101315 \rangle,$
$\langle 4111213 \rangle,$	$\langle 4111214 \rangle,$	$\langle 4111315 \rangle,$	$\langle 56912 \rangle,$	$\langle 561213 \rangle,$
$\langle 561314 \rangle,$	$\langle 57815 \rangle,$	$\langle 571011 \rangle,$	$\langle 571115 \rangle,$	$\langle 581314 \rangle,$
$\langle 67815 \rangle,$	$\langle 671011 \rangle,$	$\langle 671115 \rangle,$	$\langle 68912 \rangle,$	$\langle 681215 \rangle,$
$\langle 781314 \rangle,$	$\langle 791214 \rangle,$	$\langle 791314 \rangle,$	$\langle 891011 \rangle,$	$\langle 8101114 \rangle.$